Calculate the limit
https://www.linkedin.com/groups/8313943/8313943-6426330311789023233
$\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!}-\sqrt[3n]{n!}\right)\sqrt[3]{n^2}$.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let $a_n := \ln\left(\dfrac{\sqrt[3n+3]{(n+1)!}}{\sqrt[3n]{n!}}\right)$. Since $\lim_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}=\dfrac{1}{e}$ then $\sqrt[n]{n!}$ is asymptotically equivalent to $\dfrac{n}{e}$ $\left(\sqrt[n]{n!}\sim\dfrac{n}{e}\right)$ then $\lim_{n\to\infty}\dfrac{\sqrt[3n+3]{(n+1)!}}{\sqrt[3n]{n!}}=\lim_{n\to\infty}\dfrac{\sqrt[3]{\dfrac{n+1}{e}}}{\sqrt[3]{\dfrac{n}{e}}}=1$ implies $\lim_{n\to\infty}a_n=0$ and, therefore, $\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!}-\sqrt[3n]{n!}\right)\sqrt[3]{n^2}=$
$\lim_{n\to\infty}\sqrt[3n]{n!}\left(\dfrac{\sqrt[3n+3]{(n+1)!}}{\sqrt[3n]{n!}}-1\right)\sqrt[3]{n^2}=e^{-1/3}\lim_{n\to\infty}n\cdot\dfrac{e^{a_n}-1}{a_n}\cdot a_n=e^{-1/3}\lim_{n\to\infty}n\cdot a_n$.

We have $\lim_{n\to\infty}n\cdot a_n=\dfrac{1}{3}\lim_{n\to\infty}(3n+3)\cdot a_n=\dfrac{1}{3}\lim_{n\to\infty}\ln\left(\left(\dfrac{(n+1)!}{n!\sqrt[n]{n!}}\right)\right)=$
$\dfrac{1}{3}\ln\left(\lim_{n\to\infty}\dfrac{n+1}{\sqrt[n]{n!}}\right)=\dfrac{1}{3}\ln e=\dfrac{1}{3}$.

Thus, $\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!}-\sqrt[3n]{n!}\right)\sqrt[3]{n^2}=e^{-1/3}/3$.